Laborararatorul 5. Crearea de algoritmi proprii pentru generarea variabilelor aleatoare continue

Bibliografie:

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Scopuri:

1) Construirea in Matlab a unor functii pentru simularea unei variabile continue normale si a variabilelor inrudite.
2) Construirea unor functii in Matlab pentru simularea unei variabile continue cu o repartitie: exponentiala, Weibull, gamma.

SIMULAREA UNEI VARIABILE CU REPARTITIE NORMALA

O variabila aleatoare X are o repartitie unidimensionala $N(\mu,\sigma)$ daca admite o functie densitate de probabilitate de forma:

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

unde

$$\begin{cases} M(X)=\mu \\ D^2(X)=\sigma^2. \end{cases}$$

Functia de repartitie a variabilei aleatoare X nu are o forma analitica explicita, avand expresia urmatoare:

$$F_X(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}\mathrm{e}^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t.$$

Simularea lui X se reduce la simularea variabilei normale redusa asociata acesteia:

$$Z=\frac{X-\mu}{\sigma}\sim N(0,1).$$

Relatia precedenta poate fi justificata in urmatorul mod:

$$F_Z(z)=P(Z<z)=P\left(\frac{X-\mu}{\sigma}<z\right)=P(X<\mu+z\sigma)=F_X(\mu+z\sigma)=$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\mu+z\sigma}e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,\mathrm{d}t\overset{w=(t-\mu)/\sigma}{=}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-\frac{w^2}{2\sigma^2}}\,\mathrm{d}w=\Phi(z).$$

Intre variabilele aleatoare *X* si *Z* exista relatia:

$$X=\mu+Z\sigma.$$

Vom construi in Matlab functia repnorm.m, pentru simularea lui X:

```
function [m1,s1]=repnorm(m,sig)
z=randn(100,1);
x=m+sig*z;
m1=mean(x);
s1=std(x);
end
```

```
>>[m,s]=repnorm(1,2)
m=
    0.9155
s=
     2.0831
```

<u>Observatie</u>. Daca aplicam functia predefinita in Matlab **z=randn(n,1)** putem spune ca am simulat o selectie de volum n asupra variabilei aleatoare $Z\sim N(0,1)$.

SIMULAREA UNEI VARIABILE ALEATOARE χ^2

Fie $Z_i, 1\le i\le\gamma$ variabile normale $N(0,1)$ independente. O variabila aleatoare χ^2 cu γ grade de libertate este o variabila de forma:

$$\chi_\gamma^2=\sum_{i=1}^{\gamma}Z_i^2,\ \gamma\in N^*. \tag{1}$$

O variabila aleatoare χ_γ^2 este continua si admite densitatea de probabilitate:

$$f(x)=\frac{1}{2^{\gamma/2}\Gamma\left(\frac{\gamma}{2}\right)}\cdot x^{\frac{\gamma}{2}-1}\mathrm{e}^{-\frac{x}{2}},\ x>0,$$

unde

$$\Gamma(\gamma)=\int_0^\infty x^{\gamma-1}\mathrm{e}^{-x}\,\mathrm{d}x$$

semnifica functia Gamma a lui Euler, $\Gamma:(0,\infty)\to\Re$ si are proprietatile:

$$\begin{cases}\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\\ \Gamma(1)=1\\ \Gamma(a+1)=a\Gamma(a),\ (\forall)\,a>0\\ \Gamma(n+1)=n!,\ (\forall)\ n\in N\end{cases}$$

iar

$$\begin{cases}M\left(\chi_\gamma^2\right)=\gamma\\ D^2\left(\chi_\gamma^2\right)=2\gamma.\end{cases}$$

Pentru simularea in Matlab a unei variabile aleatoare χ_γ^2 vom folosi formula (1).

```
function x=hip(n)
z=randn(n,1);
x=sum(z.^2);
end
>> for i=1:200
x(i)=hip(10);
end
>> mean(x)
  ans =
   9.7490
>> std(x)^2
  ans =
   18.5520
```

SIMULAREA UNEI VARIABILE ALEATOARE STUDENT

Fie Z o variabila $N(0,1)$ si χ_γ^2 o variabila χ^2 cu γ grade de libertate. Presupunem ca Z si χ_γ^2 sunt independente. Numim variabila Student cu γ grade de libertate, variabila aleatoare

$$t_\gamma = \frac{Z}{\sqrt{\frac{\chi_\gamma^2}{\gamma}}}.$$

Functia densitatea de probabilitate a variabilei aleatoare t_γ este:

$$g(x) = \frac{1}{\sqrt{\pi\gamma}} \cdot \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)} \left(1 + \frac{x^2}{\gamma}\right)^{-\frac{\gamma+1}{2}}.$$

Vom construi functia Matlab care ne permite simularea variabilei aleatoare t_γ.

```
function t = stud(n)
z=randn(1,1);
h=hip(n);
t=z/sqrt(h/n);
end
>>stud(4)
ans=
    0.2128
```

SIMULAREA UNEI VARIABILE ALEATOARE SNEDECOR

Fie $\chi_{\gamma_1}^2$ si $\chi_{\gamma_2}^2$ doua variabile aleatoare χ^2 independente. Numim variabila F a lui Snedecor cu (γ_1, γ_2) grade de libertate, variabila de forma:

$$F_{\gamma_1,\gamma_2} = \frac{\gamma_2}{\gamma_1} \cdot \frac{\chi_{\gamma_1}^2}{\chi_{\gamma_2}^2}.$$

Densitatea de probabilitate a variabilei F este:

$$f(x) = \left(\frac{\gamma_1}{\gamma_2}\right)^{\frac{\gamma_1}{2}} \cdot \frac{\Gamma\left(\frac{\gamma_1+\gamma_2}{2}\right)}{\Gamma\left(\frac{\gamma_1}{2}\right)\Gamma\left(\frac{\gamma_2}{2}\right)} \cdot x^{\frac{\gamma_1}{2}-1}\left(1 + \frac{\gamma_1}{\gamma_2}x\right)^{-\frac{\gamma_1+\gamma_2}{2}},$$

iar

$$\begin{cases} M\left(F_{\gamma_1,\gamma_2}\right) = \dfrac{\gamma_2}{\gamma_2 - 2}, \gamma_2 > 2 \\ D^2\left(F_{\gamma_1,\gamma_2}\right) = \dfrac{2\gamma_2^2(\gamma_1 + \gamma_2 - 2)}{\gamma_1(\gamma_2 - 2)^2(\gamma_2 - 4)}, \ \gamma_2 \neq 2, \ \gamma_2 \neq 4. \end{cases}$$

In Matlab vom avea:

```
function [m,s] = snedec(n1,n2)
for i=1:100
h1=hip(n1);
h2=hip(n2);
F(i)=(n2/n1)*(h1/h2);
end
m=mean(F);
s=std(F)^2;
if n2>2
   mm=n2/(n2-2)
end
if n2~=2 & n2~=4
   ss=(2*n2^2*(n1+n2-2))/(n1*(n2-2)^2*(n2-4))
end
end
>> [m,s]=snedec(6,9)
mm =

   1.2857
ss =
   1.4327
m =
   1.3769
s =
   1.2140
```

SIMULAREA UNEI VARIABILE LOG- NORMALE

O variabila aleatoare X este o variabila log- normala daca $Y = \ln X$ este normala, adica are functia densitate de probabilitate:

$$f(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2},$$

unde :

$$\begin{cases} \mu_y = M(Y) = \ln \mu_x - \frac{1}{2}\ln\left[\frac{\sigma_x^2}{\mu_x^2}+1\right] \\ \sigma_y^2 = \ln\left[\frac{\sigma_x^2}{\mu_x^2}+1\right]. \end{cases}$$

Functia Matlab urmatoare simuleaza o variabila log- normala.

```
function  v=lognor(mx,sx)
spy=log(sx^2/mx^2+1);
my=log(mx)-1/2*log(sx^2/mx^2+1);
z=norm(0,1);
y=my+sqrt(spy)*z;
v=exp(y);
end
```

SIMULAREA UNEI VARIABILE CU REPARTITIE EXPONENTALA

O variabila exponentiala $X \sim Exp(\lambda)$ are functia densitatea de probabilitate:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \le 0 \end{cases}, \lambda \in \Re,$$

functia de repartitie:

$$F_X(x) = \int_{-\infty}^{x} f(t)\mathrm{d}t = \int_{0}^{x} f(t)\mathrm{d}t = 1 - e^{-\lambda x}, \quad x > 0,$$

iar

$$\begin{cases} M(X)=\dfrac{1}{\lambda} \\ D^2(X)=\dfrac{1}{\lambda^2}. \end{cases}$$

Pentru simularea unei variabile aleatoare X, ce are repartitie exponentiala vom folosi *metoda inversa*, conform careia putem simula orice variabila aleatoare X daca cunoastem functia sa de repartitie F si putem calcula functia inversa F^{-1}.

Algoritmul pentru simularea variabilei aleatoare X consta in:

- generararea unei valori u uniform repartizate in $[0,1]$,
- determinarea lui $X = F^{-1}(u) = -\frac{1}{\lambda}\ln(1-u)$.

Functia **exponential.m,** pe care am realizatat-o in Matlab permite simularea lui X cu metoda inversa:

```
function x=exponential(la)
u=rand();
x=(-1/la)*log(1-u);
end
>>for i=1:10
x(i)=exponential(1);
end
>>mean(x)
  ans=
       0.9955
>>std(x)^2
     ans=
       1.0115
```

SIMULAREA UNEI VARIABILE ALEATOARE CU REPARTITIE WEIBULL

O variabila aleatoare inrudita cu variabila exponentiala este variabila Weibull (notata $W(\alpha,\lambda,\gamma)$), a carei densitate de probabilitate este :

$$f(x)=\begin{cases} \gamma\lambda(x-\alpha)^{\gamma-1}e^{-\lambda(x-\alpha)^{\gamma}}, & x>\alpha \\ 0, & x\le\alpha \end{cases}, \alpha\in\Re,\ \lambda,\gamma>0.$$

Daca $X \sim Exp(1)$, atunci variabila Weibull se genereaza cu formula:

$$W = \alpha + \left(\frac{X}{\lambda}\right)^{\frac{1}{\gamma}}.$$

Intr-adevar, avem:

$$P(W < w) = P\left(X < \lambda(w-\alpha)^{\gamma}\right) = \int_{-\infty}^{\lambda(w-\alpha)^{\gamma}} \mathrm{e}^{-t}\,\mathrm{d}t \overset{u=\alpha+\left(\frac{t}{\lambda}\right)^{\frac{1}{\gamma}}}{=} \int_{-\infty}^{w} \gamma\lambda(u-\alpha)^{\gamma-1}\,\mathrm{e}^{-\lambda(u-\alpha)^{\gamma}}\,\mathrm{d}u.$$

Variabila Weibull se utilizează în fiabilitate, ea reprezentând durata de exploatare fără căderi a unui echipament sau produs industrial.

SIMULAREA UNEI VARIABILE ALEATOARE CU REPARTITIE GAMMA

O variabila aleatoare X are repartitia $G(\alpha,\lambda,\gamma)$ daca are functia densitate de probabilitate

$$f(x) = \begin{cases} \dfrac{\lambda^{\gamma}}{\Gamma(\gamma)}(x-\alpha)^{\gamma-1}\,\mathrm{e}^{-\lambda(x-\alpha)}, & x > \alpha \\ 0, & x \le \alpha \end{cases}$$

unde: $\alpha \in \Re,\ \lambda, \gamma > 0$ sunt respectiv parametrii de locatie, de scala si de forma ai variabilei.

Se observa ca o variabila exponentiala este variabila gamma $G(0,\lambda,1)$ iar χ^2_{γ} este o variabila gamma $G\left(0,\frac{1}{2},\frac{\gamma}{2}\right)$.

Daca $Y \sim G\left(\alpha,\lambda,\frac{\gamma}{2}\right)$ iar $Z \sim G\left(0,\frac{1}{2},\frac{\gamma}{2}\right)$ atunci avem:

$$Y = \alpha + \frac{Z}{2\lambda}. \tag{2}$$

Relatia (2) poate fi justificata in felul urmator:

$$F_Z(z) = P(Z < z) = P(2\lambda(Y-\alpha) < z) = P\left(Y < \alpha + \frac{z}{2\lambda}\right) = F_Y\left(\alpha + \frac{z}{2\lambda}\right) =$$

$$= \int_{-\infty}^{\alpha+\frac{z}{2\lambda}} \frac{\lambda^{\gamma/2}}{\Gamma\left(\frac{\gamma}{2}\right)} (t-\alpha)^{\gamma/2-1} \mathrm{e}^{-\lambda(t-\alpha)} \,\mathrm{d}t \overset{w=2\lambda(t-\alpha)}{=} \int_{-\infty}^{z} \frac{\lambda^{\gamma/2}}{\Gamma\left(\frac{\gamma}{2}\right)} \left(\frac{w}{2\lambda}\right)^{\gamma/2-1} \mathrm{e}^{-\lambda\cdot w/2\lambda} \frac{\mathrm{d}w}{2\lambda}$$

$$= \int_{-\infty}^{z} \frac{\lambda^{\gamma/2}}{2^{\gamma/2}\Gamma\left(\frac{\gamma}{2}\right)} \cdot w^{\gamma/2-1} \mathrm{e}^{-w/2} \,\mathrm{d}w.$$

Pentru simularea in Matlab a unei variabile aleatoare Y , ce are repartitia $G\left(\alpha, \lambda, \frac{\gamma}{2}\right)$ vom proceda in felul urmator: se genereaza $Z = \chi_{\gamma}^{2};$ se determina Y, utilizand (2).

```
function y =gam(al,la,n)
z=hip(n);
y=al+z/(2*la);
end
```